ON THE NORM OF A COMPOSITION OPERATOR WITH LINEAR FRACTIONAL SYMBOL

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Abstract. For any analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator C_φ is bounded on the Hardy space H^2, but there is no known procedure for precisely computing its norm. This paper considers the situation where φ is a linear fractional map. We determine the conditions under which $\|C_\varphi\|$ is given by the action of either C_φ or C_φ^* on the normalized reproducing kernel functions of H^2. We also introduce a new set of conditions on φ under which we can calculate $\|C_\varphi\|$; moreover, we identify the elements of H^2 on which such an operator C_φ attains its norm. Several specific examples are provided.

1. Introduction

For $1 \leq p < \infty$, the Hardy space H^p is the collection of all analytic functions f on $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$ with

$$\|f\|_p^p = \sup_{0<r<1} \int_0^{2\pi} \left| f\left(re^{i\theta}\right)\right|^p \frac{d\theta}{2\pi} < \infty.$$

Under this norm, H^p is a Banach space for all such p and a Hilbert space for $p = 2$. For any analytic map $\varphi : \mathbb{D} \to \mathbb{D}$, the composition operator C_φ on H^p is defined by the rule

$$C_\varphi(f) = f \circ \varphi.$$

Every composition operator is bounded, with

$$\left(\frac{1}{1-|\varphi(0)|^2}\right)^{1/p} \leq \|C_\varphi : H^p \to H^p\| \leq \left(\frac{1+|\varphi(0)|}{1-|\varphi(0)|}\right)^{1/p}. \tag{1.1}$$

These inequalities are sharp; for any value of $\varphi(0)$, there are particular examples of φ for which $\|C_\varphi\|$ equals the upper bound and examples for which $\|C_\varphi\|$ equals the lower bound. In general, though, there is no known procedure for precisely

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computing the norm of C_φ. We see from expression (1.1) that $\|C_\varphi\| = 1$ whenever $\varphi(0) = 0$. There are only a few other cases where we can determine the norm exactly; for example

(1) φ is inner; that is, $\lim_{r\uparrow 1}\left|\varphi\left(re^{i\theta}\right)\right| = 1$ for almost all θ in $[0, 2\pi)$,
(2) $\varphi(z) = az + b$ where $|a| + |b| \leq 1$,
(3) $\varphi(z) = \frac{(r+s)z+(1-s)}{r(1-s)z+(1+sr)}$ where $0 < s < 1$ and $0 \leq r \leq 1$.

These results appear in [12], [6], and [7] respectively. Cowen and MacCluer [8] provide a comprehensive treatment of this material, as well as a thorough overview of results relating to composition operators.

A straightforward argument involving Blaschke products shows that the following norm relationship holds for all $p \geq 1$:

$$\|C_\varphi : H^p \to H^p\|^p = \left\|C_\varphi : H^2 \to H^2\right\|^2.$$

Therefore it suffices to focus our attention on the Hilbert space H^2. When studying this space, it is often helpful to consider the reproducing kernel functions $\{K_w\}_{w\in\mathbb{D}}$, defined by the property that $\langle f, K_w\rangle = f(w)$ for all f in H^2. These functions have the form $K_w(z) = (1 - \overline{w}z)^{-1}$; hence

$$\|K_w\|_2 = \sqrt{\langle K_w, K_w\rangle} = \sqrt{K_w(w)} = \sqrt{\frac{1}{1 - |w|^2}}.$$

Throughout this paper, we write k_w to denote the normalized kernel function

$$k_w(z) = \frac{K_w(z)}{\|K_w\|_2} = \frac{\sqrt{1 - |w|^2}}{1 - \overline{w}z}.$$

For a subset W of $\mathbb{D}$, let $\mathcal{K}_W$ denote the closed linear span of the kernel functions $\{K_w\}_{w\in W}$. Observe that the orthogonal complement $\mathcal{K}_W^\perp$ is precisely the set of all functions in H^2 that vanish on W.

The kernel functions provide a valuable tool for the study of composition operators, in part because of the property that $C_\varphi^*(K_w) = K_{\varphi(w)}$ for any adjoint C_φ^*. Several authors have explored the connection between the kernel functions and the norm of C_φ. In the cases where we know $\|C_\varphi\|$, the norm is given by the action of the operator on the set of normalized kernel functions. This situation, however, is not true in general, a fact first proved by Appel, Bourdon, and Thrall [1].

The main results of this paper pertain to the situation where $\varphi : \mathbb{D} \to \mathbb{D}$ is a linear fractional map. In this case, we determine the conditions under which $\|C_\varphi\|$ is given by the action of either C_φ or C_φ^* on the normalized reproducing kernel functions (Theorem 4.4). We also introduce a new set of conditions on φ under which, at least in principle, we can calculate $\|C_\varphi\|$ (Theorem 5.5). For such φ, we identify the elements of H^2 on which C_φ attains its norm, each of which is a finite linear combination of kernel functions.

2. Preliminaries

Let T be a bounded operator on a Hilbert space $\mathcal{H}$. One reasonable strategy for determining $\|T\|$ is to investigate the spectrum of the operator T^*T. Since T^*T is self-adjoint, its spectral radius equals $\|T^*T\| = \|T\|^2$. The following observation underscores the connection between the spectrum of T^*T and the norm of T.

Proposition 2.1. *Let h be an element of $\mathcal{H}$; then $\|T(h)\| = \|T\|\,\|h\|$ if and only if $(T^*T)(h) = \|T\|^2\, h$.*

This proposition can be proved with a straightforward Hilbert space argument, or can be deduced from other well-known results (e.g. [10], p. 92). Whenever $\|T(h)\| = \|T\|\,\|h\|$ for $h \neq 0$, we say that the operator T *attains its norm* on the element h.

Let $\|\cdot\|_e$, $r(\cdot)$, and $r_e(\cdot)$ denote respectively the essential norm, the spectral radius, and the essential spectral radius of an operator. Here the adjective *essential* signifies that a particular quantity is taken with respect to the Calkin algebra. In light of Proposition 2.1, our next observation follows easily.

Proposition 2.2. *If $\|T\|_e < \|T\|$, then T attains its norm on an element of $\mathcal{H}$.*

Proof. Consider the positive operator T^*T; observe that

$$r_e(T^*T) = \|T^*T\|_e = \|T\|_e^2 < \|T\|^2 = \|T^*T\| = r(T^*T).$$

Therefore the largest element of the spectrum of T^*T does not belong to the essential spectrum, meaning that it is an eigenvalue of finite multiplicity. Consequently T^*T has an eigenvector corresponding to $\|T\|^2$, on which the operator T attains its norm. □

It is helpful to remember Proposition 2.2 when studying composition operators, especially since we have a formula (due to Joel Shapiro [15]) for the essential norm of C_φ on H^2. As it happens, in cases (1) and (3) where we know $\|C_\varphi\|$, the operators have the property that $\|C_\varphi\|_e = \|C_\varphi\|$, a condition sometimes called *extremal noncompactness*.

The remaining results in this section are specific to composition operators on the Hardy space H^2.

Proposition 2.3. *Suppose that the operator $C_\varphi : H^2 \to H^2$ attains its norm on an element g of H^2. If φ is not an inner function, then g cannot vanish at any point of $\mathbb{D}$.*

Proof. Suppose that $g(w) = 0$ for some w in $\mathbb{D}$. Then the function

$$\widetilde{g}(z) = \frac{g(z)}{b_w(z)} = \frac{g(z)}{\frac{w-z}{1-\overline{w}z}}$$

belongs to H^2, with $\|\widetilde{g}\|_2 = \|g\|_2$. Since φ is not an inner function, neither is the composition $b_w \circ \varphi$. Therefore

$$\lim_{r\uparrow 1} \left| \frac{g\left(\varphi\left(re^{i\theta}\right)\right)}{b_w\left(\varphi\left(re^{i\theta}\right)\right)} \right| > \lim_{r\uparrow 1} \left| g\left(\varphi\left(re^{i\theta}\right)\right) \right|$$

for θ in a set of positive measure. Hence $\|C_\varphi(\widetilde{g})\|_2 > \|C_\varphi(g)\|_2$, contradicting our choice of g. □

Corollary 2.4. *Suppose that φ is not inner; if g_1 and g_2 are functions on which C_φ attains its norm, then one is a scalar multiple of the other.*

Proof. Both g_1 and g_2 are eigenfunctions for $C_\varphi^* C_\varphi : H^2 \to H^2$ corresponding to the eigenvalue $\|C_\varphi\|^2$; moreover, $g_1(0)$ and $g_2(0)$ are both nonzero. If $g_1 - (g_1(0)/g_2(0))\, g_2$ were not identically 0, then it would be an eigenfunction corresponding to $\|C_\varphi\|^2$, in other words a function on which C_φ attains its norm, that vanishes at 0. Therefore $g_1 = (g_1(0)/g_2(0))\, g_2$, as we had hoped to show. □

We end this section with a straightforward, but remarkably useful observation. Let λ be an eigenvalue for $C_\varphi^* C_\varphi$ with a corresponding eigenfunction g; since C_φ

fixes the constant function $K_0(z) = 1$, we see that

$$\begin{aligned} g(\varphi(0)) &= \langle C_\varphi(g), K_0\rangle = \langle C_\varphi(g), C_\varphi(K_0)\rangle \\ &= \left\langle \left(C_\varphi^* C_\varphi\right)(g), K_0\right\rangle = \lambda\langle g, K_0\rangle = \lambda g(0). \end{aligned} \tag{2.1}$$

In particular, this result holds for $\lambda = \|C_\varphi\|^2$ if C_φ attains its norm on g.

3. The operator $C_\varphi^* C_\varphi$

Let

$$\varphi(z) = \frac{az+b}{cz+d}$$

be a nonconstant linear fractional self-map of $\mathbb{D}$. Cowen [6] proved that the adjoint operator C_φ^* may be written $T_\gamma C_\sigma T_\eta^*$, with

$$\begin{aligned} \sigma(z) &= \frac{\overline{a}z - \overline{c}}{-\overline{b}z + \overline{d}}, \\ \gamma(z) &= \frac{1}{-\overline{b}z + \overline{d}}, \\ \eta(z) &= cz + d, \end{aligned} \tag{3.1}$$

where T_γ and T_η denote the corresponding Toeplitz operators. Hence $\left(C_\varphi^* C_\varphi\right)(f) = \left(T_\gamma C_\sigma T_\eta^* C_\varphi\right)(f)$ for any f in H^2. Recalling that T_z^* is the backward shift on H^2, we see that

$$\begin{aligned} \left(\left(C_\varphi^* C_\varphi\right) f\right)(z) &= \gamma(z)\left(\overline{c}\left(\frac{f(\varphi(\sigma(z))) - f(\varphi(0))}{\sigma(z)}\right) + \overline{d} f(\varphi(\sigma(z)))\right) \\ &= \frac{\overline{c}}{\overline{a}z - \overline{c}}\left[f(\varphi(\sigma(z))) - f(\varphi(0))\right] + \frac{\overline{d}}{-\overline{b}z + \overline{d}} f(\varphi(\sigma(z))) \end{aligned} \tag{3.2}$$

for all z in $\mathbb{D}$ not equal to $\sigma^{-1}(0) = \frac{\overline{c}}{\overline{a}}$. We rewrite this expression simply as

$$\left(\left(C_\varphi^* C_\varphi\right) f\right)(z) = \psi(z) f(\tau(z)) + \chi(z) f(\varphi(0)), \tag{3.3}$$

where τ denotes the composition $\varphi \circ \sigma$ and

$$\psi(z) = \frac{\left(\overline{ad} - \overline{bc}\right) z}{(\overline{a}z - \overline{c})\left(-\overline{b}z + \overline{d}\right)} \text{ and } \chi(z) = \frac{\overline{c}}{-\overline{a}z + \overline{c}}.$$

Equation (3.3) holds for all points except $z = \sigma^{-1}(0)$, which only belongs to $\mathbb{D}$ if $|c| < |a|$. Having such a concrete representation for $C_\varphi^* C_\varphi$ makes it easier to investigate its spectrum.

4. The quantities S_φ and S_φ^*

Let φ be an analytic self-map of $\mathbb{D}$. Bourdon and Retsek [4] defined the quantities

$$S_\varphi = \sup_{w\in\mathbb{D}} \left\{ \frac{\|C_\varphi(K_w)\|_2}{\|K_w\|_2} \right\} = \sup_{w\in\mathbb{D}} \left\{ \|C_\varphi(k_w)\|_2 \right\}$$

and

$$S_\varphi^* = \sup_{w\in\mathbb{D}} \left\{ \frac{\left\|C_\varphi^*(K_w)\right\|_2}{\|K_w\|_2} \right\} = \sup_{w\in\mathbb{D}} \left\{ \left\|C_\varphi^*(k_w)\right\|_2 \right\}.$$

Among other results, they proved that $S_\varphi^* \le S_\varphi$ for all φ and that $S_\varphi^* = S_\varphi = \|C_\varphi\|$ whenever $\varphi(0) = 0$ or φ has the form $\varphi(z) = az + b$; moreover, when $\varphi(0) \ne 0$ and $\varphi(z) \ne az + b$, they showed that S_φ^* cannot equal $\|C_\varphi\|$ unless $\|C_\varphi\|_e = \|C_\varphi\|$. The quantities S_φ and S_φ^* were also studied, with different notation, by Avramidou and Jafari [2]. In this section, we determine the conditions under which either $S_\varphi = \|C_\varphi\|$ or $S_\varphi^* = \|C_\varphi\|$ when $\varphi : \mathbb{D} \to \mathbb{D}$ is a linear fractional map.

We begin with a few observations which hold for any analytic $\varphi : \mathbb{D} \to \mathbb{D}$. If $\{w_j\}$ is a sequence of points converging to w in $\mathbb{D}$, then the normalized kernel functions $\left\{k_{w_j}\right\}$ converge to k_w in the norm of H^2. Therefore, since C_φ is a bounded operator, either $S_\varphi = \|C_\varphi(k_w)\|_2$ for a particular w in $\mathbb{D}$ or $S_\varphi = \limsup_{|w|\uparrow 1} \|C_\varphi(k_w)\|_2$. The analogous result holds for S_φ^*. Cima and Matheson [5] observed that

$$\|C_\varphi\|_e = \limsup_{|w|\uparrow 1} \|C_\varphi(k_w)\|_2,$$

a fact which follows from the proof of Shapiro's essential norm formula [15]. In the case where φ is univalent, Shapiro's formula may be expressed

$$\|C_\varphi\|_e = \limsup_{|w|\uparrow 1} \sqrt{\frac{1-|w|^2}{1-|\varphi(w)|^2}} = \limsup_{|w|\uparrow 1} \left\|C_\varphi^*(k_w)\right\|_2.$$

Therefore $S_\varphi \ge \|C_\varphi\|_e$ for any φ and $S_\varphi^* \ge \|C_\varphi\|_e$ whenever φ is univalent.

Before proving our results for linear fractional φ, we need the following pair of general lemmas. The first, which pertains to the lower bound in expression (1.1), appears with a different proof in a current paper of David Pakorny and Jonathan Shapiro [13].

Lemma 4.1. *If $\varphi : \mathbb{D} \to \mathbb{D}$ is a nonconstant analytic map with $\varphi(0) \ne 0$, then*

$$\|C_\varphi\| > \sqrt{\frac{1}{1-|\varphi(0)|^2}}.$$

Proof. The Hardy space H^2 has the property that $|f(0)| < \|f\|_2$ for any nonconstant element f. Observe that the function $k_{\varphi(0)} \circ \varphi$ is nonconstant; therefore

$$\|C_\varphi\| \geq \left\|C_\varphi\left(k_{\varphi(0)}\right)\right\|_2 > \left|\left(k_{\varphi(0)} \circ \varphi\right)(0)\right| = \sqrt{\frac{1}{1-|\varphi(0)|^2}},$$

as we had hoped to show. □

Lemma 4.2. *Suppose that $\varphi : \mathbb{D} \to \mathbb{D}$ is a nonconstant analytic map with $\varphi(0) \neq 0$. If the operator C_φ attains its norm on a normalized kernel function k_w, then $|w| > |\varphi(0)|$.*

Proof. Suppose that C_φ attains its norm on k_w; then K_w is an eigenfunction for $C_\varphi^* C_\varphi$ corresponding to $\|C_\varphi\|^2$. Appealing to equation (2.1), we see that

$$\frac{1}{1-\overline{w}\varphi(0)} = K_w(\varphi(0)) = \|C_\varphi\|^2 K_w(0) = \|C_\varphi\|^2.$$

It follows from Lemma 4.1 that

$$\frac{1}{1-\overline{w}\varphi(0)} > \frac{1}{1-|\varphi(0)|^2},$$

meaning that $|w| > |\varphi(0)|$. □

Now we turn our attention to the situation where φ is a linear fractional map.

Proposition 4.3. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a linear fractional map with $\varphi(0) \neq 0$ and which does not have the form $\varphi(z) = az + b$. For any point w in $\mathbb{D}$,*

$$\|C_\varphi\| > \|C_\varphi(k_w)\|_2.$$

Proof. Suppose, to the contrary, that C_φ attains its norm on some normalized kernel function k_w; then K_w is an eigenfunction for $C_\varphi^* C_\varphi$. Hence the subspace $\mathcal{K}_{\{w\}} = \{\alpha K_w : \alpha \in \mathbb{C}\}$ is invariant under $C_\varphi^* C_\varphi$. Since $C_\varphi^* C_\varphi$ is self-adjoint, the orthogonal complement $\mathcal{K}_{\{w\}}^\perp = \{f \in H^2 : f(w) = 0\}$ is also invariant under the operator; this observation will give rise to a contradiction. Lemma 4.2 tells us that w cannot equal 0 or $\varphi(0)$. Suppose then that w is the problematic point $\sigma^{-1}(0) = \frac{\overline{c}}{\overline{a}}$. Applying L'Hôpital's rule to expression (3.2), we obtain

$$\left(C_\varphi^* C_\varphi(f)\right)(\sigma^{-1}(0)) = \frac{\overline{c}}{\overline{a}} f'(\varphi(0))\tau'(\sigma^{-1}(0)) + \frac{\overline{ad}}{\overline{ad}-\overline{bc}} f(\varphi(0)),$$

which must equal 0 for any f in $\mathcal{K}_{\{w\}}^\perp$. Consider the function $f_1(z) = (z-\varphi(0))(z-w)$ in $\mathcal{K}_{\{w\}}^\perp$. The assumption that $\varphi(z) \neq az + b$ guarantees that $c \neq 0$; since

$f_1(\varphi(0)) = 0$ and $\tau = \varphi \circ \sigma$ is univalent, the term $f_1'(\varphi(0))$ must equal 0, which is not the case. Therefore w cannot equal $\sigma^{-1}(0)$. Hence equation (3.3) is valid at w, meaning that

$$0 = \left(C_\varphi^* C_\varphi(f)\right)(w) = \psi(w) f(\tau(w)) + \chi(w) f(\varphi(0))$$

for all f in $\mathcal{K}_{\{w\}}^\perp$. Again consider the function f_1. Observe that $f_1(\varphi(0)) = 0$; since $w \neq 0$, the term $\psi(w)$ is nonzero. Hence $f_1(\tau(w)) = 0$, meaning that $\tau(w)$ equals either w or $\varphi(0)$. If $\tau(w) = \varphi(0)$, then $w = \sigma^{-1}(0)$, which is not the case; therefore $\tau(w) = w$. Now take $f_2(z) = z - w$ in $\mathcal{K}_{\{w\}}^\perp$. Since $f_2(\tau(w)) = f_2(w) = 0$ and $\chi(w) = \frac{\overline{c}}{\overline{c} - \overline{a}w} \neq 0$, we see that $f_2(\varphi(0)) = 0$. Therefore $\varphi(0)$ must equal w, which is a contradiction. □

We now state main result of this section:

Theorem 4.4. *Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a linear fractional map with $\varphi(0) \neq 0$ and which does not have the form $\varphi(z) = az + b$. Then $S_\varphi = \|C_\varphi\|$ if and only if $\|C_\varphi\|_e = \|C_\varphi\|$; likewise $S_\varphi^* = \|C_\varphi\|$ if and only if $\|C_\varphi\|_e = \|C_\varphi\|$.*

Proof. Recall that $\|C_\varphi\|_e \leq S_\varphi^* \leq S_\varphi \leq \|C_\varphi\|$ for any univalent φ; if $\|C_\varphi\|_e = \|C_\varphi\|$, then all of these quantities are equal. On the other hand, suppose that $\|C_\varphi\|_e < \|C_\varphi\|$. Since $\|C_\varphi(k_w)\|_2 < \|C_\varphi\|$ for all w in $\mathbb{D}$, it follows from our characterization of S_φ that $S_\varphi < \|C_\varphi\|$. Since $S_\varphi^* \leq S_\varphi$, our result follows. □

As a consequence of this theorem, we see that $S_\varphi = \|C_\varphi\|$ if and only if $S_\varphi^* = \|C_\varphi\|$. We should mention, though, that there are linear fractional φ such that $S_\varphi^* = S_\varphi < \|C_\varphi\|$; for example, Retsek [14] showed that the map $\varphi(z) = \frac{4}{5-z}$ has this property.

Theorem 4.4 no longer holds if we eliminate the hypothesis that φ be linear fractional. In light of the aforementioned results of Bourdon and Retsek, we see that our assertion for S_φ^* holds whenever φ is univalent (an observation also made by Retsek [14]). On the other hand, Bourdon and Retsek [4] proved that $S_\varphi^* < \|C_\varphi\|_e = \|C_\varphi\|$ whenever φ is a non-univalent inner function with $\varphi(0) \neq 0$. Extremal noncompactness implies that $S_\varphi = \|C_\varphi\|$ for any φ. It is not difficult, however, to find further examples of analytic φ with $\varphi(0) \neq 0$ and $\|C_\varphi\|_e < S_\varphi = \|C_\varphi\|$. To that end, let ν be an inner function that fixes the origin; then (as shown by Nordgren [12]) the composition operator C_ν is an isometry of H^2. Hence, for any analytic $\varphi : \mathbb{D} \to \mathbb{D}$, the operator $C_{\varphi\circ\nu} = C_\nu C_\varphi$ has the same norm as C_φ; moreover, $S_{\varphi\circ\nu} = \|C_{\varphi\circ\nu}\|$ if and only if $S_\varphi = \|C_\varphi\|$. Consider the map $\varphi(z) = az + b$, where both a and b are nonzero and $|a| + |b| < 1$. We know that $S_\varphi = \|C_\varphi\|$, and that both of the operators C_φ and $C_{\varphi\circ\nu}$ are compact. Hence $\|C_{\varphi\circ\nu}\|_e = 0 < S_{\varphi\circ\nu} = \|C_{\varphi\circ\nu}\|$; in particular, this result holds if we take $\nu(z) = z^m$ for some integer $m \geq 1$, so that $(\varphi \circ \nu)(z) = az^m + b$.

5. The spectrum of $C_\varphi^* C_\varphi$

Let $\varphi : \mathbb{D} \to \mathbb{D}$ be a nonconstant linear fractional map, as discussed in Section 3; let σ be defined as in (3.1). Our goal now is to find a set of conditions under which we can determine $\|C_\varphi\|$. For a non-negative integer j, let τ_j denote the jth iterate of $\tau = \varphi \circ \sigma$; that is, τ_0 is the identity map on $\mathbb{D}$ and $\tau_{j+1} = \tau \circ \tau_j$. Throughout the next two sections, we make the following assumption:

There is some integer $n \geq 0$ such that $\tau_n(\varphi(0)) = 0$.

In effect, this condition is a generalization of the case where $\varphi(0) = 0$. To avoid a triviality, we also assume that φ does not have the form $\varphi(z) = az$. These assumptions guarantee that $\tau_j(\varphi(0))$ never equals $\sigma^{-1}(0)$ and that $\tau_j(\varphi(0)) \neq 0$ for $j \neq n$, as we can see from arguments involving fixed points. Furthermore, these conditions exclude all disk automorphisms and all maps of the form $\varphi(z) = az + b$.

Let W denote the set of points $\{\tau_j(\varphi(0))\}_{j=0}^n$; recall that $\mathcal{K}_W^\perp$ is the subspace of H^2 consisting of all functions that vanish on W. We claim that $\mathcal{K}_W^\perp$ is invariant under the operator $C_\varphi^* C_\varphi$. Suppose that f belongs to $\mathcal{K}_W^\perp$; it follows from equation (3.3) that

$$\begin{aligned}\left(C_\varphi^* C_\varphi(f)\right)(\tau_j(\varphi(0))) &= \psi(\tau_j(\varphi(0)))f(\tau_{j+1}(\varphi(0))) + \chi(\tau_j(\varphi(0)))f(\varphi(0)) \\ &= \psi(\tau_j(\varphi(0)))f(\tau_{j+1}(\varphi(0))).\end{aligned}$$

For $0 \leq j \leq n-1$, the term $f(\tau_{j+1}(\varphi(0)))$ equals 0; for $j = n$, the term $\psi(\tau_j(\varphi(0))) = \psi(0)$ vanishes. Therefore $\left(C_\varphi^* C_\varphi\right)(f)$ also belongs to $\mathcal{K}_W^\perp$.

Since $C_\varphi^* C_\varphi : \mathcal{K}_W^\perp \to \mathcal{K}_W^\perp$ "looks like" a weighted composition operator, we can deduce a good deal of information about its spectrum. For example, if $C_\varphi : H^2 \to H^2$ is compact, then the spectrum of $C_\varphi^* C_\varphi : \mathcal{K}_W^\perp \to \mathcal{K}_W^\perp$ is precisely

$0 \cup \{\psi(w_0)(\tau'(w_0))^j\}_{j=0}^{\infty}$, where w_0 denotes the Denjoy-Wolff point of τ. This fact, however, does not help us to determine $\|C_\varphi\|$ and is not proved here; details appear in the author's thesis [9].

Now consider $\mathcal{K}_W$, the span of the kernel functions $\{K_{\tau_j(\varphi(0))}\}_{j=0}^{n}$; observe that it has dimension $n+1$. The subspace $\mathcal{K}_W$ is also invariant under the self-adjoint operator $C_\varphi^* C_\varphi : H^2 \to H^2$. Our strategy for finding $\|C_\varphi\|$ centers around determining the spectrum, namely the eigenvalues, of the operator $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$.

The next several results pertain to the eigenvalues and eigenfunctions of $C_\varphi^* C_\varphi$. The following proposition serves as a generalization of equation (2.1).

Proposition 5.1. *Let λ be an eigenvalue of $C_\varphi^* C_\varphi : H^2 \to H^2$ with a corresponding eigenfunction g. For every integer $j \geq 0$, the following relationship holds:*

$$\lambda^{j+1} g(0) = \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))\right] g(\tau_j(\varphi(0))) + \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j-k} g(0),$$

where we take $\prod_{m=0}^{-1}(\cdot)$ to equal 1 and $\sum_{k=0}^{-1}(\cdot)$ to equal 0.

Proof (by induction). Since $\lambda g(0) = g(\varphi(0))$, the claim holds for $j = 0$. For any $j \geq 0$, equation (3.3) dictates that

$$\begin{aligned} \lambda g(\tau_j(\varphi(0))) &= \left(\left(C_\varphi^* C_\varphi\right) g\right)(\tau_j(\varphi(0))) \\ &= \psi(\tau_j(\varphi(0))) g(\tau_{j+1}(\varphi(0))) + \chi(\tau_j(\varphi(0))) \lambda g(0). \end{aligned} \tag{5.1}$$

Now assume that our claim holds for the index j. Multiplying the consequent equation by λ and substituting expression (5.1) for $\lambda g(\tau_j(\varphi(0)))$, we obtain

$$\begin{aligned} \lambda^{j+2} g(0) &= \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))\right] \left[\psi(\tau_j(\varphi(0))) g(\tau_{j+1}(\varphi(0))) + \chi(\tau_j(\varphi(0))) \lambda g(0)\right] \\ &\quad + \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j+1-k} g(0) \\ &= \left[\prod_{m=0}^{j} \psi(\tau_m(\varphi(0)))\right] g(\tau_{j+1}(\varphi(0))) \\ &\quad + \sum_{k=0}^{j} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j+1-k} g(0). \end{aligned}$$

Hence our claim also holds for the index $j+1$. □

Since both $\mathcal{K}_W$ and $\mathcal{K}_W^{\perp}$ are invariant under $C_\varphi^* C_\varphi : H^2 \to H^2$, each eigenvalue λ of $C_\varphi^* C_\varphi$ has an eigenfunction belonging to one of the two subspaces. The next proposition provides a distinguishing characteristic for eigenfunctions in $\mathcal{K}_W^{\perp}$.

Proposition 5.2. *Let g be an eigenfunction for $C_\varphi^* C_\varphi : H^2 \to H^2$; then g belongs to $\mathcal{K}_W^{\perp}$ if and only if $g(0) = 0$.*

Proof. If g belongs to $\mathcal{K}_W^{\perp}$, then by definition $g(0) = g(\tau_n(\varphi(0)))$ equals 0. Conversely, suppose that g is an eigenfunction for $C_\varphi^* C_\varphi$ with $g(0) = 0$. In this case, Proposition 5.1 dictates that

$$0 = \lambda^{j+1} g(0) = \left[\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0))) \right] g(\tau_j(\varphi(0)))$$

for all $j \geq 0$. Since $\psi(\tau_m(\varphi(0)))$ is nonzero for $0 \leq m \leq n-1$, the function g must vanish on the entire set $\{\tau_j(\varphi(0))\}_{j=0}^{n}$. In other words, g belongs to the subspace $\mathcal{K}_W^{\perp}$. □

Corollary 5.3. *Suppose that g_1 and g_2 are eigenfunctions for $C_\varphi^* C_\varphi$ which belong to $\mathcal{K}_W$; if they correspond to the same eigenvalue, then one is a scalar multiple of the other.*

Proof. We appeal to the proof of Corollary 2.4, bearing in mind that no eigenfunction in $\mathcal{K}_W$ can vanish at 0. □

Consequently every eigenspace of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ has dimension 1. Since $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ is a self-adjoint operator on a finite dimensional space, we know that $\mathcal{K}_W$ is spanned by eigenfunctions of $C_\varphi^* C_\varphi$. Since $\mathcal{K}_W$ has dimension $n+1$, the operator $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ must have $n+1$ distinct eigenvalues.

We return to the result of Proposition 5.1. Taking $j = n$ and observing that $\chi(\tau_n(\varphi(0))) = \chi(0) = 1$, we obtain the expression

$$\lambda^{n+1} g(0) = \sum_{k=0}^{n} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0))) \right] \lambda^{n-k} g(0).$$

Suppose that the eigenfunction g belongs to $\mathcal{K}_W$; then $g(0) \neq 0$ and

$$\lambda^{n+1} = \sum_{k=0}^{n} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0))) \right] \lambda^{n-k}.$$

In other words, any eigenvalue λ of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ is a solution to this polynomial equation. Since there are $n+1$ distinct eigenvalues and the equation has no more than $n+1$ roots, we conclude that every solution is an eigenvalue. In other words,

$$p(\lambda) = \lambda^{n+1} - \sum_{k=0}^{n} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0))) \right] \lambda^{n-k} \tag{5.2}$$

is the characteristic polynomial of the operator $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$.

Finally, we make an observation regarding the essential norm of C_φ. (The author is indebted to Paul Bourdon for suggesting the proof of this proposition.)

Proposition 5.4. *Under the assumptions of this section,* $\|C_\varphi\|_e < 1$.

Proof. If $\|\varphi\|_\infty < 1$, then C_φ is compact, so our claim holds. Suppose then that $\|\varphi\|_\infty = 1$; since φ is not an automorphism, there is precisely one pair of points ζ and ω on $\partial\mathbb{D}$ with $\varphi(\zeta) = \omega$. Bourdon, Levi, Narayan, and Shapiro [3] proved in general that $\sigma(\omega) = \zeta$ and $\sigma'(\omega) = (\varphi'(\zeta))^{-1}$; hence $\tau(\omega) = \omega$ and $\tau'(\omega) = 1$. Since the map $\tau_n \circ \varphi$ fixes the origin and $(\tau_n \circ \varphi)(\zeta) = \omega$, it follows from Lemma 7.33 in [8], together with the Julia-Carathéodory theorem, that $|(\tau_n \circ \varphi)'(\zeta))| > 1$. Therefore

$$1 < |(\tau_n)'(\varphi(\zeta)) \cdot \varphi'(\zeta)| = |(\tau_n)'(\omega) \cdot \varphi'(\zeta)| = |\varphi'(\zeta)|.$$

Since φ is univalent on a neighborhood of the closed unit disk, Shapiro's essential norm formula [15] yields

$$\|C_\varphi\|_e^2 = \max\left\{ |\varphi'(w)|^{-1} : |w| = |\varphi(w)| = 1 \right\} = |\varphi'(\zeta)|^{-1} < 1,$$

as we had hoped to show. ☐

Since $\|C_\varphi\| \geq 1$, Proposition 2.2 dictates that $C_\varphi : H^2 \to H^2$ attains its norm on an element of H^2; that is, $\|C_\varphi\|^2$ is an eigenvalue of $C_\varphi^* C_\varphi : H^2 \to H^2$. Proposition 2.3 guarantees that any corresponding eigenfunction must belong to $\mathcal{K}_W$. In other words, $\|C_\varphi\|^2$ is the largest eigenvalue of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$, meaning that it is the largest zero of the polynomial p. Hence we have proved the following result:

Theorem 5.5. *Let* $\varphi : \mathbb{D} \to \mathbb{D}$ *be a linear fractional map, with* $\varphi(z) \neq az$. *Suppose that* $\tau_n(\varphi(0)) = 0$ *for some integer* $n \geq 0$*; then* $\|C_\varphi\|^2$ *is the largest zero of the* *polynomial p in equation (5.2), and the elements on which C_φ attains its norm are linear combinations of the kernel functions* $\left\{K_{\tau_j(\varphi(0))}\right\}_{j=0}^{n}$.

Whenever $n \geq 1$, Theorem 4.4 dictates that $S_\varphi < \|C_\varphi\|$. Assuming that we can find examples of such φ, this would appear to be the first case of a composition operator whose norm we can calculate, for which the norm is not given by the action of C_φ on the normalized reproducing kernel functions of H^2.

6. The eigenfunctions of $C_\varphi^* C_\varphi$

Having determined a particular eigenvalue λ of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$, it is possible to find the corresponding eigenfunctions. In particular, considering Theorem 5.5, we can identify the functions on which the operator C_φ attains its norm. Let λ be such an eigenvalue and g be its unique eigenfunction in $\mathcal{K}_W$ with $g(0) = g(\tau_n(\varphi(0))) = 1$. We write

$$g(z) = \sum_{i=0}^{n} \frac{\alpha_i}{1 - \overline{\tau_i(\varphi(0))}z},$$

where we hope to determine the coefficients α_i. For any index $0 \leq j \leq n-1$, we may appeal to Proposition 5.1 to find $g(\tau_j(\varphi(0)))$ explicitly in terms of λ:

$$g(\tau_j(\varphi(0))) = \frac{\lambda^{j+1} - \sum_{k=0}^{j-1} \chi(\tau_k(\varphi(0))) \left[\prod_{m=0}^{k-1} \psi(\tau_m(\varphi(0)))\right] \lambda^{j-k}}{\prod_{m=0}^{j-1} \psi(\tau_m(\varphi(0)))}.$$

Therefore we obtain the matrix equation

$$\left[\frac{1}{1 - \overline{\tau_i(\varphi(0))}\tau_j(\varphi(0))}\right]_{0\leq j,i\leq n} [\alpha_i]_{0\leq i\leq n} = [g(\tau_j(\varphi(0)))]_{0\leq j\leq n}.$$

The $(n+1)\times(n+1)$ matrix is simply the Gram matrix of the vectors $\left\{K_{\tau_i(\varphi(0))}\right\}_{i=0}^{n}$, whose determinant is positive since the kernel functions are linearly independent (see [11], p. 595). Hence we can use Cramer's rule to solve explicitly for the coefficients.

For example, take $n = 1$. Then

$$\begin{bmatrix} \frac{1}{1-|\varphi(0)|^2} & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix} = \begin{bmatrix} g(\varphi(0)) \\ g(0) \end{bmatrix} = \begin{bmatrix} \lambda \\ 1 \end{bmatrix},$$

so

$$\alpha_0 = \frac{\begin{vmatrix} \lambda & 1 \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \frac{1}{1-|\varphi(0)|^2} & 1 \\ 1 & 1 \end{vmatrix}} = \frac{\lambda - 1}{\frac{1}{1-|\varphi(0)|^2} - 1} \text{ and } \alpha_1 = \frac{\begin{vmatrix} \frac{1}{1-|\varphi(0)|^2} & \lambda \\ 1 & 1 \end{vmatrix}}{\begin{vmatrix} \frac{1}{1-|\varphi(0)|^2} & 1 \\ 1 & 1 \end{vmatrix}} = \frac{\frac{1}{1-|\varphi(0)|^2} - \lambda}{\frac{1}{1-|\varphi(0)|^2} - 1}.$$

7. Examples

It is not difficult to find examples of linear fractional $\varphi : \mathbb{D} \to \mathbb{D}$ with $\tau(\varphi(0)) = 0$. In terms of the coefficients of φ, this condition is equivalent to

$$|d|^2 - |b|^2 = \frac{a}{b}\left(\bar{c}d - \bar{a}b\right).$$

In this case, considering the polynomial p in equation (5.2), we see that any eigenvalue λ of $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ has the form

$$\lambda = \frac{\chi(\varphi(0)) \pm \sqrt{\chi(\varphi(0))^2 + 4\psi(\varphi(0))}}{2} = \frac{\frac{a\bar{c}d}{b} \pm \sqrt{\left(\frac{a\bar{c}d}{b}\right)^2 - 4\left(\overline{ad} - \overline{bc}\right)ad}}{2\left(|d|^2 - |b|^2\right)}.$$

In particular, $\|C_\varphi\|^2$ is the larger of these two values. For example, take

$$\varphi(z) = \frac{16z + 8}{19z + 32}.$$

Since $\|\varphi\|_\infty < 1$, the operator C_φ is compact. Observe that $\tau(\varphi(0)) = 0$, which means that

$$\|C_\varphi\|^2 = \frac{19 + \sqrt{181}}{30} \approx 1.081787468.$$

We now turn our attention to a larger class of examples. Let n be a positive integer and r a real number greater than n. Define

$$\varphi(z) = \frac{rz - n}{-(n+1)z + (r+1)}.$$

It is easy to show that φ is a self-map of $\mathbb{D}$ and that $\partial\varphi(\mathbb{D}) \cap \partial\mathbb{D} = \{1\}$. Note that

$$\|C_\varphi\|_e^2 = |\varphi'(1)|^{-1} = \frac{(r-n)^2}{r(r+1) - n(n+1)} = \frac{r-n}{r+n+1}.$$

A straightforward induction argument shows that each iterate τ_j has the form

$$\tau_j(z) = \frac{(r + n - j + 1)\, z + j}{-jz + (r + n + j + 1)}.$$

Consequently

$$\tau_j(\varphi(0)) = \frac{(r+n-j+1)\left(-\frac{n}{r+1}\right)+j}{-j\left(-\frac{n}{r+1}\right)+(r+n+j+1)} = \frac{j-n}{r+j+1},$$

from which we see that $\tau_n(\varphi(0)) = 0$. Observe that

$$\begin{aligned}\psi(\tau_j(\varphi(0))) &= \frac{(r(r+1)-n(n+1))\left(\frac{j-n}{r+j+1}\right)}{\left(r\left(\frac{j-n}{r+j+1}\right)+n+1\right)\left(n\left(\frac{j-n}{r+j+1}\right)+r+1\right)} \\ &= \frac{(r-n)(j-n)(r+j+1)}{(j+1)(r+n+1)(r+j-n+1)}\end{aligned}$$

and

$$\chi(\tau_j(\varphi(0))) = \frac{n+1}{r\left(\frac{j-n}{r+j+1}\right)+n+1} = \frac{(n+1)(r+j+1)}{(j+1)(r+n+1)}.$$

Hence the characteristic polynomial for $C_\varphi^* C_\varphi : \mathcal{K}_W \to \mathcal{K}_W$ may be written

$$p(\lambda) = \lambda^{n+1} - \sum_{k=0}^{n} \frac{(n+1)(r+k+1)}{(k+1)(r+n+1)}\left[\prod_{m=0}^{k-1} \frac{(r-n)(m-n)(r+m+1)}{(m+1)(r+n+1)(r+m-n+1)}\right]\lambda^{n-k},$$

and $\|C_\varphi\|^2$ is the largest zero of this polynomial.

In particular, if $n = 1$ then

$$\|C_\varphi\|^2 = \frac{r+1}{r+2} + \frac{1}{r+2}\sqrt{\frac{2(r+1)}{r}}.$$

For $n = 2$, we solve the resulting cubic equation to obtain

$$\|C_\varphi\|^2 = \frac{r+1}{r+3} + \frac{2}{r+3}\sqrt[3]{\frac{3(r+1)}{r(r-1)}}\,\mathrm{Re}\left(\sqrt[3]{(r+4)+i(r-2)\sqrt{\frac{2(r+2)}{r-1}}}\right),$$

where we take the principal branch of the cube root function. For example, if

$$\varphi(z) = \frac{4z-2}{-3z+5}$$

then

$$\|C_\varphi\|^2 = \frac{5}{7} + \frac{2\,\mathrm{Re}\,\sqrt[3]{10+5i}}{7} = \frac{5}{7} + \frac{2\sqrt{5}}{7}\cos\left(\frac{\arctan\left(\frac{1}{2}\right)}{3}\right) \approx 1.345547525.$$

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